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Islands in three-dimensional steady flows

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We consider the problem of steady Euler flows in a torus. We show that in the absence of a direction of symmetry the solution for the vorticity contains δ-function singularities at the rational surfaces of the torus. We study the effect of a small but finite viscosity on these singularities. The solutions near a rational surface contain cat's eyes or islands, well known in the classical theory of critical layers. When the islands are small, their widths can be computed by a boundary-layer analysis. We show that the islands at neighbouring rational surfaces generally overlap. Thus, steady toroidal flows exhibit a tendency towards Beltramization.

1. Introduction

The existence and structure of steady solutions of the Euler equations of fluid flow have been the subject of interesting papers by Arnol'd (1974) and Moffatt (1985, 1986). In these papers, the analogy between the Euler equations for steady flow of an inviscid, incompressible fluid and the equations for magnetostatic equilibria of a perfectly conducting plasma provides the basis for certain deductions on the fundamental properties of three-dimensional Euler flows. Invoking the principle of relaxation of magnetic fields to states of minimum energy (well known in plasma physics, and first stated by Kruskal & Kulsrud 1958), Moffatt has shown, by means of topological arguments in certain special examples, how current sheets may develop in ideal plasmas. By analogy, vortex sheets may develop in Euler flows. Moffatt has motivated these studies with the speculation that 'insofar as viscous effects may be neglected, Euler flows may be regarded as fixed points in the function space in which unsteady solutions of the Euler equations evolve, and, even if these fixed points are unstable, their location in function space may provide valuable clues concerning the structure of turbulent flow' (Moffatt 1985).

In this paper, we consider steady, three-dimensional flows in a torus without a continuous symmetry. Since the subject of this paper was inspired in part by the analogy between magnetostatics and Euler flows, it is natural to consider the problem of Euler flows in the geometry of a torus. The existence and structure of magnetostatic equilibria has been an important subject of research for plasma physicists over two decades (Grad 1967), and it is interesting to examine the implications of certain known results on toroidal plasma equilibria for analogous Euler flows (Hegna & Bhattacharjee 1990). It is known that if a direction of symmetry exists, the magnetostatic equilibria are characterized by a set of nested toroidal surfaces. On each surface, the plasma pressure is constant and the magnetic field line is constrained to remain on the toroidal surface. If the symmetry is lost, the equilibria contain current sheets. The analogous three-dimensional Euler flows contain vortex sheets.

Apart from the analogy between magnetostatics and Euler flows, there is a deeper physical reason which makes the study of *toroidal* Euler flows interesting in its own right. The reason for considering toroidal Euler flows is that the vortex singularities have a simple spatial structure and occur at rational (or resonant) surfaces of the torus, which is where our intuition on dynamical systems leads us to expect them. In fact, once the appropriate representation for the velocity $\boldsymbol{u}$ is written down, it is straightforward to give a rigorous demonstration of the existence of singularities in the component of the vorticity $\boldsymbol{\omega}(\equiv \nabla \times \boldsymbol{u})$ aligned with $\boldsymbol{u}$. These vortex singularities can be represented by δ-functions (§2).

While the proof of existence of the vortex singularities settles an important point of principle, the presence of a small but finite viscosity has a profound effect on these vortex sheets. In the presence of viscosity, the singularities in the vorticity are resolved, and their amplitudes can be determined by a boundary-layer analysis of the steady Navier–Stokes equation. The boundary-layer method we use is reminiscent of the nonlinear theory of critical layers in which the presence of viscosity (or nonlinearity) can regularize a singularity in the exterior-region solution (Benney & Bergeron 1969; Haberman 1972). However, the geometry of a torus introduces certain interesting features not present in earlier analyses. One of the main features is that the toroidal problem is intrinsically three-dimensional, which naturally brings in issues pertaining to the non-integrability of the Hamiltonian for the velocity field.

The analytical solutions contain cat's eyes (hereafter referred to as islands) at rational surfaces. If the islands are not too large, their widths can be computed under certain simplifying approximations (§3). In the neighbourhood of a particular resonant surface, we solve the steady-state equation under the assumption that the effect of other non-resonant terms may be neglected. This approximation, which is standard in most perturbative studies of nearly integrable Hamiltonian systems, is valid if the internal separatrices of the torus are sufficiently well separated in space. A similar approximation has been used recently by Childress & Soward (1989) in an interesting study of a family of cat's eye flows.

Our asymptotic expression for the island width near a resonant surface leads us to a striking conclusion regarding the topological structure of generic steady three-dimensional toroidal flows. We show that it is generally impossible to avoid overlap of islands on neighbouring rational surfaces in a torus (§4). Though the destruction of KAM (Kolmogorov–Arnol'd–Moser) surfaces cannot be described within our mathematical framework, Chirikov's criterion of stochasticity (Chirikov 1979) suggests that ergodization of streamlines (and vortex lines) will occur under these conditions.

These results were reported briefly in our recent contribution to the Proceedings of the IUTAM Symposium held in Cambridge (United Kingdom), 13–18 August, 1989 (Hegna & Bhattacharjee 1990). Readers interested in a comparative description of our results on magnetostatic equilibria and Euler flows are referred to that paper. The details of the calculation of three-dimensional magnetostatic equilibria in toroidal geometry have been published previously (Hegna & Bhattacharjee 1989). Here we give the details of the analysis which led to the conclusions on steady Euler flows reported at the IUTAM Symposium.

2. Vortex singularities in steady toroidal Euler flows

Steady flows of an inviscid, incompressible fluid of constant density are described by the equations

$$\boldsymbol{u} \times \boldsymbol{\omega} = \nabla h, \tag{1}$$

$$\nabla \times \boldsymbol{u} = \boldsymbol{\omega}, \tag{2}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{3}$$

where $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{\omega}$ is the vorticity, and $h = p/\rho + \frac{1}{2}|\boldsymbol{u}|^2$ is the Bernoulli function for a fluid of pressure p and constant density ρ. Equations (1)–(3) are also the equations of magnetostatics in which the analogous variables are the magnetic field, $\boldsymbol{B} \leftrightarrow \boldsymbol{u}$, the current density, $\boldsymbol{J} \leftrightarrow \boldsymbol{\omega}$, and the plasma pressure $p \leftrightarrow h_0 - h$, where h_0 is a constant. Kruskal & Kulsrud (1958) have derived a number of useful properties of (1)–(3). In particular, if h is a smooth function and not constant in any small region, the local surfaces of h determine a family of surfaces which according to (1), are 'stream (vortex) surfaces' in that they are made up of 'stream (vortex) lines'. If such a surface lies in a bounded volume of space, and if either $\boldsymbol{u}$ or $\boldsymbol{\omega}$ vanishes nowhere on it, then it must be topologically toroidal (Kruskal & Kulsrud 1958; Moffatt 1988).

We now introduce a curvilinear coordinate system that has been used to represent magnetic fields in magnetostatic equilibria (Greene & Johnson 1961; Boozer 1981). In order to keep this paper self-contained, we derive this representation from first principles in Appendix A. The nested surfaces are labelled radially by a (single-valued) function Φ such that $\boldsymbol{u} \cdot \nabla \Phi = 0$. For convenience, we take Φ to be a measure of the toroidal flux of $\boldsymbol{u}$. Each surface is parameterized by a poloidal angle θ and a toroidal angle ϕ which increases by 2π for each transit of a streamline in the poloidal and toroidal directions respectively. In the coordinate system (Φ, θ, ϕ), the velocity $\boldsymbol{u}$ can be represented in the contravariant form

$$\boldsymbol{u} = \nabla \Phi \times \nabla(\theta - \iota \phi), \tag{4}$$

where $\iota = \iota(\Phi)$ is the rotational transform of the velocity field. The rotational transform measures the twist of the streamline and is an invariant on each stream surface. We assume that ι is a smooth function of Φ. Since ι is continuous, the set of values Φ for which ι is rational is of measure zero. There are closed field lines near every point in the domain, but the set of points that are actually on closed field lines has zero measure. From (1) and (4), it then follows that $h = h(\Phi)$. Equation (4) is reminiscent of the well known Clebsch representation for divergence-free vector fields. Alternatively, a covariant basis can also be used to represent $\boldsymbol{u}$ in the form,

$$\boldsymbol{u} = g(\Phi)\, \nabla \phi + I(\Phi)\, \nabla \theta + \beta(\Phi, \theta, \phi)\, \nabla \Phi. \tag{5}$$

This representation satisfies identically $\boldsymbol{\omega} \cdot \nabla h = 0$, required by (1). The Jacobian $\mathcal{J} \equiv (\nabla \Phi \cdot \nabla \theta \times \nabla \phi)^{-1}$ can be obtained from the scalar product of (4) and (5),

$$\mathcal{J} = \frac{g + \iota I}{u^2} = \frac{\gamma(\Phi)}{u^2}. \tag{6}$$

All single-valued functions of position in the torus can be expressed as a Fourier sum. In particular,

$$\mathcal{J} = \sum_{m,n} \mathcal{J}_{mn}(\Phi)\, \mathrm{e}^{\mathrm{i}m\theta - \mathrm{i}n\phi}, \tag{7}$$

where m and n are integers. From (1), we get

$$\boldsymbol{\omega}_{\perp} = \frac{\nabla h \times \boldsymbol{u}}{u^2}, \tag{8}$$

where the subscript $\perp$ refers to the direction perpendicular to $\boldsymbol{u}$. Since $h = h(\Phi)$, we get, using (5) and (6),

$$\boldsymbol{\omega}_{\perp} = \frac{\mathcal{J}}{\gamma} h' \nabla \Phi \times (g \nabla \theta + I \nabla \phi), \tag{9}$$

where prime denotes derivative with respect to Φ. We write the vorticity aligned with $\boldsymbol{u}$ as

$$\omega_{\parallel} = Q\boldsymbol{u}, \tag{10}$$

where Q is represented by the Fourier series

$$Q = \sum_{m,n} Q_{mn}(\Phi)\, \mathrm{e}^{\mathrm{i}m\theta-\mathrm{i}n\phi}. \tag{11}$$

Equation (2) implies $\nabla \cdot \omega = 0$, which is a relation between $\omega_{\perp}$ and $\omega_{\parallel}$. This relation gives

$$(\iota m - n)\, Q_{mn} = \frac{h'}{\gamma}(mg + nI)\, \mathscr{J}_{mn}. \tag{12}$$

The amplitude Q_{00} cannot be determined from (12), but can be calculated by fixing an additional global constraint, such as the total toroidal flux of the vorticity. The general solution of (12) is given by

$$Q_{mn} = h' \mathscr{J}_{mn}\left(\frac{1}{\iota - n/m} - \frac{I}{\gamma}\right) + \hat{Q}_{mn}\, \delta(\Phi - \Phi_{\mathrm{r}}), \tag{13}$$

where $\Phi = \Phi_{\mathrm{r}}$ at the resonant (or rational) surface $\iota = n/m$. At a rational surface, the streamlines do not fill a surface ergodically, but close on themselves after m poloidal and n toroidal transits. The presence of the singularity in Q_{mn} can be anticipated by noting that the differential equation $\boldsymbol{u} \cdot \nabla Q = -\nabla \cdot \omega_{\perp}$ is singular at a rational surface because the operator $\boldsymbol{u} \cdot \nabla$ is not invertible at such a surface.

Equation (13) shows that Q, and hence $\omega_{\parallel}$, is singular at a rational surface. Singularities appear in the first and third terms of (13), but both singularities are integrable. The amplitude $\hat{Q}_{mn}$ of the δ-function singularity cannot be determined from the local inversion of (1) described above. Additional physical constraints, specified in §3, determine $\hat{Q}_{mn}$.

Before concluding this section, we remark on the possibility of generating the singular solutions (13) by the 'magnetic relaxation' method (Kruskal & Kulsrud 1958; Bauer, Batancourt & Garabedian 1978; Moffatt 1988). If the initial state is a magnetic field (without symmetry) which lies on topologically toroidal surfaces, the final equilibrium state generated by the relaxation method should contain singularities of the type considered in this paper. The strength of the singularity can be determined, in principle, from the initial conditions. A physically meaningful way of specifying initial conditions is to specify certain invariants of motion.

The relaxation method has been numerically implemented in calculating three-dimensional toroidal magnetostatic equilibria by plasma physicists (Bauer *et al.* 1978; Hirshman & Whitson 1983; Bhattacharjee, Wiley & Dewar 1984; Lao *et al.* 1985). In practice, it is difficult to track singularities in a computer code even when it is known where they are expected to occur! In most computer codes, the presence of 'numerical' dissipation or truncation errors regularize singular solutions (see, for instance, Lao *et al.* 1985).

3. Resolution of the vortex singularities

Though the formation of vortex singularities in an ideal fluid is a matter of fundamental interest, in practice the presence of even a small but finite viscosity will regularize these singularities. Far from a rational surface ('exterior region'), what appears to be a δ-function singularity is resolved by the effect of a small viscosity which makes its presence felt in the close neighbourhood of the rational surface ('interior region'). The appropriate mathematical tool to use is boundary-layer theory in which $\omega_{\parallel}$ in the exterior region is given by (13), and the interior region is localized at a particular rational surface where we solve (1) by a different method. The asymptotic matching between the exterior and interior regions determines $\hat{Q}_{mn}$, the amplitude of the δ-function singularity in $\omega_{\parallel}$.

Before we proceed with the details of the boundary-layer analysis, we illustrate the formation of islands by means of a simple example. Consider a straight periodic cylinder. An integrable velocity field is given by

$$\boldsymbol{u}_0 = u_\theta(r)\,\hat{\theta} + u_z(r)\,\hat{\zeta}, \tag{14}$$

where the azimuthal velocity u_θ and the axial velocity u_z are functions of the radius r only, $\zeta = z/R$, and $2\pi R$ is the periodicity length of the cylinder. The equation for the streamline is given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \boldsymbol{u}_0, \tag{15}$$

where τ is a time-like coordinate which parameterizes points along the streamline. If $u_z \neq 0$, it is convenient to choose ζ as the time-like coordinate. Equation (15) can then be written

$$\frac{\mathrm{d}r}{\mathrm{d}\zeta} = \frac{\boldsymbol{u}_0\cdot\nabla r}{\boldsymbol{u}_0\cdot\nabla\zeta} = 0, \tag{16a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{\boldsymbol{u}_0\cdot\nabla\theta}{\boldsymbol{u}_0\cdot\nabla\zeta} = \Omega, \tag{16b}$$

where $\Omega = Ru_\theta/ru_z$ is a function of r only. The streamlines lie on surfaces of constant r. On a given surface the twisting of a streamline is given by $\theta = \Omega\zeta + \theta_0$, for some initial condition $\theta(\zeta = 0) = \theta_0$.

Now suppose a symmetry-breaking field $\boldsymbol{u}_1 = \boldsymbol{u}_1(r, m\theta - n\zeta)$ is imposed on the velocity field given by (14), with the radial projection of $\boldsymbol{u}_1$ given by

$$u_r = u_{mn}\sin(m\theta - n\zeta). \tag{17}$$

Note that in the presence of this perturbation the system is still integrable because an ignorable coordinate remains. If $|\boldsymbol{u}_1| \ll |\boldsymbol{u}_0|$, a new function that can label stream surfaces can be obtained from perturbation theory. We seek functions χ such that $\boldsymbol{u}\cdot\nabla\chi = 0$, where

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1, \quad \chi = r + \sum_n \epsilon^n \chi_n,$$

and $\epsilon \equiv |\boldsymbol{u}_1|/|\boldsymbol{u}_0|$. To first order in ϵ, we get

$$\boldsymbol{u}_0\cdot\nabla\chi_1 + u_r = 0. \tag{18}$$

Equation (18) gives

$$\chi = r + \frac{Ru_{mn}/mu_z}{\Omega - n/m}\cos(m\theta - n\zeta) + O(\epsilon^2), \tag{19}$$

if $\Omega \neq n/m$. Away from the rational surfaces, the stream surfaces are slightly distorted but remain nested. However, near $\Omega = n/m$, the small denominator creates large excursions in χ, and the expansion given above is no longer valid. To rectify this problem, we must solve for χ by a different method. Consider the streamlines when $u_{1\theta} \ll u_\theta$ and $u_{1z} \ll u_z$; we get

$$\frac{\mathrm{d}r}{\mathrm{d}\zeta} = \frac{u_{mn}\sin(m\theta - n\zeta)}{u_z}, \tag{20a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \Omega. \tag{20b}$$

If the time-like coordinate is redefined as $\tau = \theta - n\zeta/m$, then

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{u_{mn}\sin(m\tau)}{u_\theta/r - nu_z/mR}. \tag{21}$$

It follows that streamlines must obey the condition

$$\int \mathrm{d}r \frac{u_z}{u_{mn}}(\Omega - n/m) = R\int \mathrm{d}\tau \sin(m\tau). \tag{22}$$

Suppose that at some value of $r(r = r_0)$ in the cylinder, $\Omega(r_0) = n/m$. In the vicinity of $r = r_0$, assume u_{mn}/u_z varies slowly, whence (22) can be written as

$$\int \mathrm{d}r\, \Omega'(r - r_0) = \frac{Ru_{mn}}{u_z}\int \mathrm{d}\tau \sin(m\tau), \tag{23}$$

where $\Omega' = \mathrm{d}\Omega/\mathrm{d}r$ at $r = r_0$. Upon integrating this equation, the trajectories of the streamline near $r = r_0$ are found to lie on level surfaces of a new function Ξ, given by

$$\Xi = \tfrac{1}{2}\Omega'(r - r_0)^2 + \frac{Ru_{mn}}{mu_z}\cos(m\tau). \tag{24}$$

A Poincaré plot in the (r, θ)-plane at $\zeta = 0$ formed by successive transits of a streamline in a periodic cylinder shows that initial points given by $r = r_0$, $\sin(m\tau) = 0$ are period-m fixed points. Near $r = r_0$, $\cos(m\tau) = -1$, for $\Omega' > 0$ trajectories execute elliptic orbits around the local axis of the island labelled by $\Xi_a = -Ru_{mn}/mn_z$, whereas the points $r = r_0$, $\cos(m\tau) = 1$ are hyperbolic fixed points. The surface $\Xi_x = Ru_{mn}/mu_z$ describes the separatrix. The island half-width is given by

$$w_{mn} = 2(Ru_{mn}/m\Omega' u_z)^{\frac{1}{2}}. \tag{25}$$

In the presence of incommensurate helical perturbations, islands grow at different rational surfaces. Roughly speaking, if these islands are large enough to overlap, no stream function exists. In this paper, the occurrence of overlapping islands will be taken to imply that the velocity field lines are chaotic in the domain (Chirikov 1979).

The qualitative features of the simple example given above will reappear in the boundary-layer analysis that follows. Note that in (17), the perturbation is imposed arbitrarily without any consideration of how the fluid may generate such a perturbation. In the toroidal analysis to follow, the perturbation is self-consistent in that it obeys the steady Navier–Stokes equation.

3.1. *The exterior region*

In the torus, we consider the formation of a single island at the surface $ɫ = n_r/m_r$. It is convenient to transform to the new angle coordinates

$$\alpha = \theta - (n_r/m_r)\phi, \tag{26}$$

$$\zeta = \phi. \tag{27}$$

We introduce a new function ψ in the representation of the velocity field $\boldsymbol{u}$, which is written as

$$\boldsymbol{u} = \nabla\Phi \times \nabla\alpha + \nabla\zeta \times \nabla\psi. \tag{28}$$

The function ψ is represented by the Fourier series

$$\psi = \psi_0(\Phi) - \sum_{m,n} A_{mn}(\Phi)\, e^{im\alpha + i[m(n_r/m_r) - n]\zeta}, \tag{29}$$

where $d\psi_0/d\Phi = \iota - n_r/m_r$. We are primarily concerned with the Fourier component of ψ resonant with α. We define the average

$$\bar{f} \equiv (2\pi)^{-1} \int_0^{2\pi} d\zeta f(\Phi, \alpha, \zeta), \tag{30}$$

for any function f. At the resonant surface defined by $\iota(\Phi_r) = n_r/m_r$, the singularity in the vorticity causes a discontinuity in the exterior solution of $\bar{A}$, measured by the parameter

$$\Delta' = \frac{\partial_\Phi \bar{A}(\Phi_r^+) - \partial_\Phi \bar{A}(\Phi_r^-)}{\bar{A}(\Phi_r)}. \tag{31}$$

As it stands now, Δ' includes all values of m and n such that $n/m = n_r/m_r$. (Eventually, we will use the single-harmonic approximation, in which the amplitude corresponding to $m = m_r, n = n_r$ dominates over all others.) By integrating (2), $\nabla \times \boldsymbol{u} = \boldsymbol{\omega}$, across a resonant surface, we can obtain a relation between Δ' and the singularity amplitude $\hat{Q}_{mn}$. Using the representation (28) in (2), and projecting (2) along $\boldsymbol{e}_\zeta = \partial \boldsymbol{x}/\partial\zeta$, we note that the dominant contribution to the left-hand side of (2) comes from the second derivative of ψ with respect to Φ. Hence, we get

$$\nabla\Phi \cdot \nabla\Phi\, \partial_\Phi^2 A = \boldsymbol{\omega} \cdot \boldsymbol{e}_\zeta, \tag{32}$$

where $\partial_x f \equiv \partial f/\partial x$. Integrating (32) across the rational surface, and averaging, we get

$$\Delta' \bar{A} = \bar{G}\bar{\hat{Q}}, \tag{33}$$

where

$$\bar{\hat{Q}} = \sum_k \hat{Q}_{km_r, kn_r} e^{ikm_r\alpha}, \tag{34}$$

and

$$G = \left(\frac{\gamma}{\nabla\Phi \cdot \nabla\Phi}\right)_{\Phi = \Phi_r}. \tag{35}$$

3.2. *The interior region*

As in the simple example at the beginning of this section, we shall obtain an expression for the invariant function ψ which is valid in the interior region. This can be done by a straightforward application of perturbation theory, neglecting the coupling between the islands on different rational surfaces (Cary & Littlejohn 1983; Cary & Kotschenreuther 1985; Hegna & Bhattacharjee 1989). The result is

$$\psi = \int (\iota - n_r/m_r)\, d\Phi - \sum_k A_k(\Phi)\, e^{ikm_r\alpha}, \tag{36}$$

which, upon using the single-harmonic approximation, reduces to

$$\psi = \tfrac{1}{2}\iota' x^2 - A(\Phi)\cos(m_r \alpha), \tag{37}$$

where $x = \Phi - \Phi_r$ and $A(\Phi) \equiv \tfrac{1}{2}[A_1(\Phi) + A_{-1}(\Phi)]$. The dominant structure in the interior region is the cat's eye or island. The island half-width in terms of the variable Φ is

$$\delta\Phi = 2|A_x/\iota'|^{\frac{1}{2}}, \tag{38}$$

where A_x is the value of A at the separatrix, and we have neglected the variation of A with Φ over the island width. By normalizing $\delta\Phi$ to the local shear length $1/\iota'$, the island half-width in terms of the extent of the rotational transform is

$$\delta\iota = 2|A_x\iota'|^{\frac{1}{2}}. \tag{39}$$

Viscosity enters the analysis of the interior region via the steady-state Navier–Stokes equation, which can be written as

$$\boldsymbol{u}\times\boldsymbol{\omega} = \boldsymbol{\nabla}h - \nu\nabla^2\boldsymbol{u}, \tag{40}$$

where ν is the coefficient of kinematic viscosity. The projection of (40) along $\boldsymbol{u}$ gives

$$\boldsymbol{u}\cdot\boldsymbol{\nabla}h = \nu(\nabla^2 h - \omega^2). \tag{41}$$

Integrating (41) over a volume bounded by streamlines, we get the solubility condition

$$\nu\int \mathrm{d}\tau\,(\nabla^2 h - \omega^2) = 0. \tag{42}$$

Equation (42) is a statement of the conservation of energy in steady state (see, for instance, Landau & Lifshitz 1986). Since the flow is steady, the time-rate of change of the kinetic energy must vanish. Hence, energy dissipation within the volume must be balanced by energy flux through the boundary. Since the volume is bounded by streamlines, and no transfer of fluid mass into the volume can occur, the energy flux due to internal friction must balance the viscous dissipation.

We now proceed with the analysis of the interior region, in which the functions h and Q are determined from the boundary-layer equations. It is convenient to introduce the bracket, defined by

$$[B,C] \equiv \partial_\Phi B\,\partial_\alpha C - \partial_\Phi C\,\partial_\alpha B. \tag{43}$$

The algebraic properties of the bracket are discussed in Appendix B. The condition $\boldsymbol{u}\cdot\boldsymbol{\nabla}h = 0$ which strictly holds for Euler flows, also holds approximately for Navier–Stokes flows as long as ν is small. (Formally, one might imagine writing a perturbation expansion for each undetermined function in powers of the viscosity; the above condition is then the leading-order approximation to the projection of the Navier–Stokes equation along $\boldsymbol{u}$.) Hence, we get

$$\partial_\zeta h + [\psi, h] = 0. \tag{44}$$

In the small-island approximation, h is a function of Φ and α alone. Therefore, (44) implies that $h = h(\psi)$. The perpendicular vorticity,

$$\boldsymbol{\omega}_\perp = \frac{\boldsymbol{\nabla}h\times\boldsymbol{u}}{u^2} \tag{45}$$

(where $u = |\boldsymbol{u}|$), and the parallel vorticity,

$$Q = \frac{\boldsymbol{u}\cdot\boldsymbol{\omega}}{u^2} \tag{46}$$

are related to each other by the equation $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$, whence

$$(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,Q = \boldsymbol{u}\cdot\boldsymbol{\nabla}h\times\boldsymbol{\nabla}(u^{-2}). \tag{47}$$

Averaging over the angle ζ, and noting that $\boldsymbol{u}\cdot\boldsymbol{e}_\zeta = \gamma$ and $\gamma/u^2 = \mathscr{J}$, we can write (47) in the form

$$[\psi, Q - h'\bar{\mathscr{J}}] = 0. \tag{48}$$

Equation (48) has the general solution (Appendix B)

$$Q = h'\bar{\mathscr{J}} + f(\psi), \tag{49}$$

where $f(\psi)$ is an as yet undetermined function. To determine the function $f(\psi)$, we now use the solubility condition (42). Though the technical details are somewhat different, the procedure is similar to that of Benney & Bergeron (1969). The integral in (42) is taken over a shell bounded by flux surfaces ψ and $\psi+\delta\psi$. We get

$$\int_0^{2\pi} \mathrm{d}\zeta \int_0^{2\pi} \mathrm{d}\alpha \int_{\psi}^{\psi+\delta\psi} \mathrm{d}\psi \frac{\mathscr{J}}{\partial_\Phi \psi}(\nabla^2 h - \omega^2) = 0. \tag{50}$$

Since the integrand is nearly constant across the infinitesimal thickness $\delta\psi$, the integrand over ψ is trivial. Defining the angle average

$$\langle * \rangle = \frac{\oint \mathrm{d}\alpha \dfrac{\bar{*}}{\partial_\Phi \psi}}{\oint \dfrac{\mathrm{d}\alpha}{\partial_\Phi \psi}}, \tag{51}$$

(50) reduces to

$$\langle Q^2 \rangle = \left\langle \frac{\mathscr{J}}{\gamma}\nabla^2 h \right\rangle - \left\langle \left(\frac{\mathscr{J}}{\gamma}\right)^2 \nabla h \cdot \nabla h \right\rangle. \tag{52}$$

From (49) and (52), it is possible to determine the function $f(\psi)$, and hence Q, the parallel vorticity profile. The result is

$$Q = h'(\psi)\mathscr{J}_0'(x - \langle x \rangle) \pm \{\langle (\mathscr{J}/\gamma)\nabla^2 h\rangle - (h'\mathscr{J}_0')^2[\langle x^2\rangle(1+p) - \langle x\rangle^2]\}^{\frac{1}{2}}. \tag{53}$$

Here, we have approximated $\bar{\mathscr{J}}$ by the Taylor-series expression $\bar{\mathscr{J}} \approx \mathscr{J}_0 + \mathscr{J}_0' x$, and $\mathscr{J}_0$ where $\mathscr{J}_0'$ are, respectively, the values of $\bar{\mathscr{J}}$ and its derivative with respect to Φ at the rational surface. The term p in (53) is given by

$$p = (\mathscr{J}_0 \imath'/\gamma\mathscr{J}_0')^2 |\nabla\Phi|^2. \tag{54}$$

Equation (53) can be simplified further if we assume that the island width is small. Formally, we introduce a small parameter $\lambda \equiv \delta\Phi/\Phi_0 \ll 1$, where $\delta\Phi$ is the island half-width and Φ_0 represents a characteristic equilibrium scale. We also define the dimensionless quantity $\beta \equiv 2h/u^2$, where u is a typical value of the equilibrium flow, and h is the spatially varying part of Bernoulli's function. In (53), the following ordering holds: $\mathscr{J}_0, \mathscr{J}_0', \gamma$, and p are $O(1)$, $\langle x \rangle$ and x are $O(\lambda)$, and h' is $O(\beta/\lambda)$. From this ordering, we see that in (53), the first term under the square root scales as $(\beta/\lambda)^{\frac{1}{2}}$, whereas all other terms scale as β. As long as $(\beta\lambda)^{\frac{1}{2}} \ll O(1)$, the term proportional to $\langle \nabla^2 h \rangle^{\frac{1}{2}}$ dominates all other terms. (If we take the ordering $\beta\lambda \sim O(1)$ or $\beta\lambda \gg O(1)$, it can be shown, *a posteriori*, that contradictions with the approximation $\lambda \ll 1$ occur.) Hence, (53) simplifies to

$$Q \approx \left(\frac{\mathscr{J}_0}{\gamma}\langle \nabla^2 h \rangle\right)^{\frac{1}{2}}, \tag{55}$$

where the negative root is neglected because it leads to unphysical solutions for the island width, obtained in §4.

The inequality $(\beta\lambda)^{\frac{1}{2}} \ll 1$ may seem to allow for the possibility that β can be large for $\lambda \ll 1$. However, as we demonstrate in §4, the requirement that $\lambda \ll 1$ imposes the further constraint that β itself be small.

The h-profile in the inner region can be determined in general by specifying the sources at steady state, not considered here. When there are no sources of pressure within the island region, the h-profile is approximately constant within the island separatrix. Far from the island, the h-profile must, of course, match smoothly to the exterior profile. Our qualitative conclusions do not depend on the global details of the h-profile.

We now carry out the asymptotic matching procedure between the exterior and interior solutions. In the interior region the matching parameter Δ' is defined by the relation

$$\Delta' = \frac{\partial_\Phi A(+\infty) - \partial_\Phi A(-\infty)}{A(\Phi_r)}. \tag{56}$$

Under the assumption that the island width is small ($\lambda \ll 1$), (2) reduces to

$$\partial_\Phi^2 A = GQ, \tag{57}$$

where G is defined by (35). Using the single-harmonic approximation, and neglecting the variation of A over the width of the island, we get

$$\Delta' A(\Phi_r) = \frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\alpha \cos(m_r \alpha) \int_{-\infty}^{+\infty} \mathrm{d}\Phi\, GQ. \tag{58}$$

From (55)–(58), we obtain

$$\Delta' A = (\delta \mathfrak{t})^{\frac{1}{2}} (|D_F k|)^{\frac{1}{2}}, \tag{59}$$

where

$$D_F = \frac{h' G \mathscr{I}_0}{\mathfrak{t}'}, \tag{60}$$

is a dimensionless function, and k is a definite integral of order unity.

4. Island equation

The amplitude of the island can be obtained analytically by solving (2) in the exterior region. We do so under the approximation that the stream surfaces are nearly circular in cross-section, and can therefore be labelled by the cylindrical radius r. This approximation, which is reasonable for low-β fluids, decouples the different helicities and enables us to write (2) as

$$\left(\frac{\partial}{\partial r}\frac{r/R}{1+\left(\frac{n_r r}{m_r R}\right)^2}\frac{\partial}{\partial r} - \frac{m_r^2}{rR}\right) A_{m_r n_r} = \frac{\mathrm{d}h}{\mathrm{d}r}\frac{\mathscr{I}_{m_r n_r}}{\mathfrak{t} - \frac{n_r}{m_r}} + \hat{Q}_{m_r n_r}\,\delta(r - r_{m_r n_r}), \tag{61}$$

where R is the length of the cylinder and $r = r_{m_r n_r}$ fixes the position of the rational surface. Equation (61) can be solved by using the appropriate Green's function solution (Cary & Kotschenreuther 1985),

$$G(r, r') = -\frac{n^2 r r'}{m^2 R} I'_m(nr_</R)\, K'_m(nr_>/R), \tag{62}$$

where I_m and K_m are the modified Bessel functions, $r_< = \min(r, r')$, and $r_> =$

max (r, r'). The integrals obtained can be evaluated asymptotically, and the resonant amplitude is given by

$$A_{m_r n_r} = \frac{C}{\iota'} + \frac{R\hat{Q}_{m_r n_r}}{2m_r}, \tag{63}$$

where
$$C \approx \left| \frac{\mathrm{d}h}{\mathrm{d}r} \frac{R^2}{u^2 r} \frac{\mathscr{J}_{m_r n_r}}{\mathscr{J}_{00}} \right|_{\Phi = \Phi_r}. \tag{64}$$

As in the simple example, the island half-width, written in units of the rotational transform, is given by

$$\delta\iota = 2\,|A_{m_r n_r} \iota'|^{\frac{1}{2}}. \tag{65}$$

From (63)–(65), we obtain the algebraic relation

$$(\delta\iota)^2 = 4C + (\delta\iota)^{\frac{1}{2}} \frac{q}{m_r}, \tag{66}$$

where $q = |D_F k|^{\frac{1}{2}} |\mathrm{d}\ln\iota/\mathrm{d}r|$. We now note that the term C in (66) scales with the resonant Jacobian amplitude which, for low-β systems, decays exponentially rapidly with mode numbers m and n (Cary & Kotschenreuther 1985). Neglecting C in (66), we obtain

$$\delta\iota = (q/m_r)^{\frac{2}{3}}. \tag{67}$$

We check, *a posteriori*, that the term C is of order $(\beta)^{\frac{1}{3}}(\mathscr{J}_{m_r n_r}/\mathscr{J}_{00})$ smaller than the other terms in (66) and hence subdominant.

The scaling of $\delta\iota$ with m has an important consequence for the topological stability of steady toroidal flows. The mean density of islands is given by $\mathrm{d}N/\mathrm{d}\iota \approx \frac{1}{2}M^2$ for islands with mode number $m_r < M$ (Cary & Kotschenreuther 1985). Island overlap will occur when $\delta\iota(\mathrm{d}N/\mathrm{d}\iota)$ exceeds $2/\pi$ (Chirikov 1979). From (67), it follows that for generic flows overlap of the islands with $m = 1$ to M is unavoidable, and will occur for values of M which obey the condition $M \gtrsim q^{-\frac{1}{2}}$. (Note that island overlap would not necessarily have occurred if $\delta\iota$ decayed exponentially with m_r.) Though the analysis presented here clearly breaks down when island overlap actually occurs, Chirikov's criterion leads us to believe that ergodization of the streamlines will occur under these conditions. From the relation $\boldsymbol{u}\cdot\boldsymbol{\nabla}h = 0$, we know that the ergodization of streamlines will tend to eliminate spatial gradients in h. Hence, it appears that the Beltrami state $\boldsymbol{\nabla}\times\boldsymbol{u} = \sigma\boldsymbol{u}$ with constant σ is the only physically realizable steady low-β flow in a three-dimensional torus (without symmetry). This intrinsic fragility, which we associate with island overlap, is consistent with Moffatt's conjecture that 'a three-dimensional flow of any significant complexity is in general unstable', though the word 'instability' should be interpreted a little differently in our context.

5. Discussion

In this paper, we have considered the problem of steady-state solutions to the Euler equations in a torus without a direction of continuous symmetry. For a configuration with nested toroidal surfaces, we show that the component of vorticity $\boldsymbol{\omega}$ parallel to the velocity $\boldsymbol{u}$ contains δ-function singularities at the rational surfaces. After demonstrating the occurrence of tangential discontinuities in a three-dimensional toroidal equilibrium, we have considered the effect of a small but finite viscosity on these singularities. By means of a boundary-layer analysis, subject to a few simplifying assumptions, we have obtained an expression for the island width when it is small compared with the characteristic equilibrium scale length. The expression for the island width leads to the striking conclusion that the islands inevitably overlap for generic flows. Since we expect that stochasticity will occur when islands overlap, the condition $\boldsymbol{u}\cdot\nabla h = 0$ then requires that h be a constant over the region of stochasticity. Thus, steady three-dimensional toroidal flows, when they are physically realizable, would tend to be Beltrami-like.

The results of our study are not directly applicable to the problem of finite-time singularities in solutions of the time-dependent Euler equations. Our study of the steady-state problem does indicate that one should look for such singularities at the separatrices of the flow. In a torus these separatrices have a very simple structure, which is why we find them easily. However, numerical studies of three-dimensional fluid turbulence involve flows of considerable topological complexity, and the identification of the separatrices in such flows is a non-trivial task. Even after the separatrices of the flow are identified, there remains the difficulty of numerically resolving the singularities. This continues to be an area of ongoing research (see, for instance, Pumir & Siggia 1990 and Kerr & Hussain 1989).

As quoted earlier, Moffatt has conjectured that Euler flows with tangential discontinuities may be regarded as fixed points in the function space in which time-dependent solutions of the Euler equations evolve. Though our work says nothing about this conjecture, it does suggest that if the conjecture is true, Beltramization of the flow will tend to occur. Recent simulations of decaying isotropic turbulence seem to confirm a certain tendency to Beltramization, but this does not seem to necessarily drive the depression of the mean-square value of the nonlinear term $\{|\boldsymbol{u}\times\boldsymbol{\omega}-\nabla h|\}^2$ in the Navier–Stokes equation (Kraichnan & Panda 1988). It is possible that the mechanism of island overlap presented in this paper may Beltramize turbulent flows, even if these flows may never be strictly steady state.

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Appendix A. Coordinate system

The coordinate system used in this paper was introduced and developed by Boozer (1981, 1983) as a convenient way to represent magnetic fields in toroidal equilibria. Because of the mathematical analogy that exists between the magnetostatic equations and the Euler equations for steady flow in an incompressible, inviscid fluid, these coordinates also prove useful for carrying out the calculations in this paper.

The representations (4) and (5) can be derived as follows. Let $\boldsymbol{u}$ be a velocity field in a toroidal equilibrium that satisfies the condition $\boldsymbol{u}\cdot\nabla\Phi^* = 0$ for some single-valued function Φ^*. The angles θ^* and ϕ represent the poloidal and toroidal angles of the torus, such that at a fixed Φ^*, the pair of angle coordinates (ϕ, θ^*) and $(\phi+2\pi n, \theta^*+2\pi m)$ represent the same physical position for all integers n and m. As long as $\nabla\Phi^*\cdot\nabla\theta^*\times\nabla\phi$ remains finite, Φ^*, θ^*, and ϕ can be used as coordinates (Boozer 1981). Any vector field can be written as

$$\boldsymbol{u} = a\nabla\Phi^*\times\nabla\theta^* + b\nabla\phi\times\nabla\Phi^* + c\nabla\theta^*\times\nabla\phi. \tag{A 1}$$

The condition $\boldsymbol{u}\cdot\nabla\Phi^* = 0$ requires $c = 0$. Incompressibility implies that a and b can be written as

$$a = a_0(\Phi^*)+a_0\frac{\partial f}{\partial\theta^*}, \quad b = b_0(\Phi^*)-a_0\frac{\partial f}{\partial\phi}, \tag{A 2}$$

where $f = f(\Phi^*, \theta^*, \phi)$ (Boozer 1981). By defining $\Phi = \int a_0\,\mathrm{d}\Phi^*$, $\psi = \int b_0\,\mathrm{d}\Phi^*$, and $\theta = \theta^*+f$, equation (A 1) can be written

$$\boldsymbol{u} = \nabla\Phi\times\nabla(\theta-\iota\phi), \tag{A 3}$$

where

$$\iota(\Phi) = \frac{\mathrm{d}\psi}{\mathrm{d}\Phi} = \frac{b_0}{a_0}, \tag{A 4}$$

is the rotational transform. The toroidal velocity flux enclosed by a constant-Φ surface is $2\pi\Phi$, and the poloidal velocity flux outside a constant-ψ surface is $2\pi\psi$.

Using Φ, $\theta^\dagger$, and $\phi^\dagger$ as the basis, a covariant representation for the velocity field can be written,

$$\boldsymbol{u} = \alpha\nabla\phi^\dagger+\gamma\nabla\theta^\dagger+\beta^\dagger\nabla\Phi. \tag{A 5}$$

If a solution to the Euler equations exists, h must be a function of Φ alone. From the condition $\nabla\times\boldsymbol{u}\cdot\nabla h = 0$, we obtain the relation

$$\frac{\partial\alpha}{\partial\theta^\dagger}-\frac{\partial\gamma}{\partial\phi^\dagger} = 0. \tag{A 6}$$

The functions α and γ can then be written in the form

$$\alpha = g(\Phi)+(g+\iota I)\frac{\partial\nu}{\partial\phi^\dagger}, \tag{A 7}$$

$$\gamma = I(\Phi)+(g+\iota I)\frac{\partial\nu}{\partial\theta^\dagger}, \tag{A 8}$$

where $\nu = \nu(\Phi, \theta^\dagger, \phi^\dagger)$. Equation (A 5) can now be written

$$\boldsymbol{u} = g(\Phi)\,\nabla\phi^\dagger+I(\Phi)\,\nabla\theta^\dagger+\left(\beta^\dagger-\frac{\partial}{\partial\Phi}[\nu(g+\iota I)]\right)\nabla\Phi+\nabla(g+\iota I)\,\nu. \tag{A 9}$$

Let $\theta = \theta^\dagger+\iota\nu$, and $\phi = \phi^\dagger+\nu$, so $\nabla\theta^\dagger = \nabla\theta-\iota\nabla\nu-\nu(\mathrm{d}\iota/\mathrm{d}\Phi)\,\nabla\Phi$ and $\nabla\phi^\dagger = \nabla\phi-\nabla\nu$. The transformation from $\theta^\dagger$, $\phi^\dagger$ to θ, ϕ does not change the representation (A 3), but allows (A 5) to be written in the form

$$\boldsymbol{u} = g\nabla\phi+I\nabla\theta+\beta(\Phi, \theta, \phi)\,\nabla\Phi, \tag{A 10}$$

where $\beta = \beta^\dagger-(g+\iota I)\,(\partial\nu/\partial\Phi)-\nu I(\mathrm{d}\iota/\mathrm{d}\Phi)$. The total toroidal vorticity inside a flux surface is given by $2\pi I$, while the poloidal vorticity outside a flux surface is $2\pi g$.

A velocity field, one that does not necessarily lie on nested surfaces, can be written as

$$\boldsymbol{u} = \nabla\Phi\times\nabla\theta+\nabla\phi\times\nabla\psi(\Phi, \theta, \phi) \tag{A 11}$$

(Boozer 1983; Bhattacharjee 1984). To demonstrate this, let V, θ, and ϕ be coordinates, where V is a label of an arbitrarily defined toroidal surface, while θ and ϕ are the poloidal and toroidal angles parameterizing the torus, with $\nabla V\cdot\nabla\theta\times\nabla\phi$ finite. An arbitrary velocity field can be written

$$\boldsymbol{u} = \frac{\partial\Phi}{\partial V}(V, \theta, \phi)\,\nabla V\times\nabla\theta+\frac{\partial\psi}{\partial V}(V, \theta, \phi)\,\nabla\phi\times\nabla V+C\nabla\theta\times\nabla\phi. \tag{A 12}$$

The condition $\nabla\cdot\boldsymbol{u} = 0$ gives

$$\frac{\partial}{\partial V}\left(\frac{\partial \Phi}{\partial \phi}(V,\theta,\phi)+\frac{\partial \psi}{\partial \theta}(V,\theta,\phi)+C\right) = 0, \tag{A 13}$$

which implies that ψ and Φ can be chosen such that $C = C_0 - \partial_\phi \Phi - \partial_\theta \psi$, where C_0 is a constant. Regularity of the velocity at the axis requires $C_0 = 0$. Therefore, the velocity field can be written in the form given by (A 11).

The Hamiltonian nature of streamline flow is transparent in the representation (A 11) (Cary & Littlejohn 1983; Dewar 1985). The trajectory of a streamline is defined by the equation

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}\times \boldsymbol{u} = 0, \tag{A 14}$$

where τ is a time-like variable that labels points along a streamline. This means that along a streamline

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} = \frac{\boldsymbol{u}\cdot\nabla\Phi}{\boldsymbol{u}\cdot\nabla\phi}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{\boldsymbol{u}\cdot\nabla\theta}{\boldsymbol{u}\cdot\nabla\phi}. \tag{A 15a, b}$$

Using the representation (A 11), the Hamilton's equations for stream-lines are

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\phi} = -\frac{\partial \psi}{\partial \theta}, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{\partial \psi}{\partial \Phi}, \tag{A 16a, b}$$

where the poloidal angle θ is the canonical coordinate, the toroidal flux Φ is the canonical momentum, the toroidal angle ϕ is the time-like coordinate and ψ is the Hamiltonian. If a symmetry exists, the Hamiltonian has one degree of freedom, and the system is integrable. The velocity field can then be written in the form given by (A 3), where the rotational transform corresponds to the frequency in the action-angle coordinates well known in classical mechanics.

Appendix B. Bracket notation

The bracket notation is introduced for mathematical convenience, with a number of useful properties that are reviewed here. The bracket is defined as

$$[A,B] = \mathscr{J}\nabla\zeta\cdot\nabla A\times\nabla B, \tag{B 1}$$

which is equivalent to

$$[A,B] = \partial_\Phi A\,\partial_\alpha B - \partial_\alpha A\,\partial_\Phi B, \tag{B 2}$$

where Φ, α and ζ define the coordinate system. $[A,B]$ behaves like a Poisson bracket, in that it is bilinear, and has the properties

$$[A,B] = -[B,A], \tag{B 3}$$

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]] = 0, \tag{B 4}$$

$$[AB,C] = A[B,C]+[A,C]B, \tag{B 5}$$

where (B 4) is the Jacobi identity, and (B 5) is the Leibniz rule. The bracket also has the property

$$\overline{[A,B]} = [\bar{A},\bar{B}]+\overline{[\tilde{A},\tilde{B}]}, \tag{B 6}$$

where

$$\bar{C} = \int\frac{\mathrm{d}\zeta}{2\pi}C, \quad \tilde{C} = C-\bar{C}. \tag{B 7}$$

The partial differential equation for f defined by

$$[\psi, f] = K, \tag{B 8}$$

where ψ and K are known functions, can be solved by integrating along the characteristics. Equation (B 8) has the solution

$$f = f_0(\psi) - \int \mathrm{d}\alpha\, K(\psi, \alpha)/\partial_{\Phi}\psi, \tag{B 9}$$

where f_0 is a free function which is a solution to the homogeneous equation $[\psi, f] = 0$.

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